INEQUALITIES FOR THE MINIMAL EIGENVALUE
OF THE LAPLACIAN IN AN ANNULUS

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ABSTRACT. We discuss the behavior of the minimal eigenvalue λ of the Dirichlet Laplacian in the domain $D_1 \backslash D_2 := D$ (an annulus) where D_1 is a circular disc and $D_2 \subset D_1$ is a smaller circular disc. It is conjectured that the minimal eigenvalue λ has a maximum value when D_2 is a concentric disc. If h is a displacement of the center of the disc D_2 and $\lambda(h)$ is the corresponding minimal eigenvalue, then $\frac{d\lambda(h)}{dh} < 0$ so that $\lambda(h)$ is minimal when ∂D_2 touches ∂D_1, where ∂D is the boundary of D. Numerical results are given to back the conjecture. Upper and lower bounds are given for $\lambda(h)$. The above conjecture is proved.

1. Introduction

Let D_1 be a disc on $\mathbf{R}^2$, centered at the origin, of radius 1, $D_2 \subset D_1$ be a disc of radius $a < 1$, the center $(h, 0)$ of which is at the distance h from the origin. Denote by $\lambda(h)$ the minimal Dirichlet eigenvalue of the Laplacian in the annulus $D := D_h := D_1 \backslash D_2$.

In this paper the following conjecture is formulated and proved:

Conjecture C. *The minimal eigenvalue $\lambda(h)$ is a monotonically decreasing function of h*

on the interval $0 \leq h \leq 1 - a$. In particular

(1.1)
$$\lambda(0) > \lambda(h), \qquad h > 0.$$

Let $\dot{\lambda} := \frac{d\lambda}{dh}$ and let S denote ∂D_2, the boundary of D_2.

The following results are given to back this conjecture:

Lemma 1. *One has*

(1.2)
$$\dot{\lambda} = \int_S u_N^2 N_1 ds,$$

where N is the unit normal to $S = S_h$ pointing into the annulus D_h, N_1 is the projection of N onto x_1-axis, u_N is the normal derivative of u, and $u(x) = u(x_1, x_2)$ is the normalized in $L^2(D)$ eigenfunction corresponding to the first eigenvalue λ:

(1.3)
$$\Delta u + \lambda u = 0 \quad in \quad D, \ u = 0 \quad on \quad \partial D_1 \cup \partial D_2 := \partial D,$$

(1.4)
$$\|u\|_{L^2(D)} = 1.$$

It is argued at the end of Section 2 that

(1.5)
$$\dot{\lambda} < 0 \quad if \quad 0 < h < 1 - a.$$

In Lemma 2 below we give upper and lower bounds (1.6) for $\lambda(h)$. These bounds are practically convenient, especially for small h.

Let $D(r)$ be the disc $|x| \leq r$, $\mu(r)$ be the first Dirichlet eigenvalue of the Laplacian in $D_1 \setminus D_1(r)$. In Section 3 inequality (1.5) is illustrated by the numerical results in $D_1 \setminus D(r)$.

Lemma 2. *One has*

(1.6)
$$\mu(a - h) < \lambda(h) < \mu(a + h), \qquad 0 < h < 1 - a, \quad h < a.$$

In section 2 proofs are given and the conjecture is proved.

2. Proofs.

Proof of Lemma 2. Lemma 2 is an immediate consequence of the variational principle for λ since $D_1 \setminus D(a + h) \subset D_h \subset D_1 \setminus D(a - h)$. Note that $\mu(b)$, $a \leq b < 1$, can be calculated

efficiently. Indeed, by symmetry the first eigenfunction ϕ of the Dirichlet Laplacian in $D_1 \backslash D(b)$ depends on the radial variable $r = |x|$ only, and solves the problem

(2.1) $$\phi'' + \frac{1}{r}\phi' + \mu\phi = 0, \qquad b \leq r \leq 1; \qquad \phi(b) = \phi(1) = 0.$$

Thus

(2.2) $$\phi = c_1 J_0(\sqrt{\mu}r) + c_2 N_0(\sqrt{\mu}r),$$

where J_0 and N_0 are the Bessel functions, and c_1, c_2 are constants. The boundary conditions (2.1) are satisfied if $\mu = \mu(b) > 0$ is a positive root of the equation:

(2.3) $$J_0(\sqrt{\mu}b)N_0(\sqrt{\mu}) - J_0(\sqrt{\mu})N_0(\sqrt{\mu}b) = 0.$$

The smallest positive root $\mu = \mu(b)$ of (2.3) is the desired first eigenvalue of the Dirichlet Laplacian in $D_1 \backslash D(b)$. Equation (2.3) can be solved numerically. This makes (1.6) an efficient estimate of $\lambda(h)$, especially for small $h > 0$.

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Proof of Lemma 1. We use the known technique based on the domain derivative [1].

It is known that $\lambda(h)$ is continuously differentiable with respect to h [2]. Let $\dot{u} = \frac{du}{dh}$, where u solves (1.3)-(1.4). Differentiate the equation and the boundary condition (1.3) with respect to h and get

(2.4) $$\Delta\dot{u} + \lambda\dot{u} = -\dot{\lambda}u \qquad in \qquad D = D_h,$$

(2.5) $$\dot{u} + u_N N_1 = 0 \ on \ S = S_h.$$

Multiply (2.4) by u, (1.3) by $\dot{u}$, subtract, integrate over $D = D_h$, use Green's formula, and (2.5) and get:

(2.6) $$\dot{\lambda} \int_D u^2 dx = \int_S (u\dot{u}_N - \dot{u}u_N)ds = \int_S u_N^2 N_1 ds.$$

From (2.6) and (1.4) one gets (1.2). Lemma 1 is proved.

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It follows from (1.2) by symmetry that $\dot{\lambda}(0) = 0$. Indeed, if $h = 0$, then $u_N^2|_{S_0} = const$ by symmetry, and $\int_{S_0} N_1 ds = 0$.

If $h > 0$, then u_N^2 on the half circle S_h^+, the part of the boundary of S_h which is closer to ∂D_1, is likely to be less than on the other half S_h^- of S_h, while $N_1 > 0$ on S_h^+ and

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$N_1 < 0$ on S_h^-. Moreover, $|N_1|$ is the same at the symmetric points of S_h^+ and S_h^-, where the axis of symmetry is the vertical diameter of D_2. Therefore one expects $\dot{\lambda}(h) < 0$ for $h > 0$, which is the conjecture C.

Let us prove that the above argument is indeed valid. What we wish to prove is the inequality for the normal derivative u_N mentioned above.

The following argument completes the proof of the Conjecture (C). This argument was communicated to AGR by Professor M.Ashbaugh. Consider the reflection of the part of the domain which is situated to the right of the vertical line passing through the center of the smaller disc with respect to this line. Let D_h denote the domain symmetric with respect to this line ℓ and v denote the function equal to u to the right of ℓ, and equal to w to the left of ℓ. Here $w(x, y) = u(x, -y)$, where the y-axis is the line ℓ. By the maximum principle one has $u > v$ on the part of the boundary of D_h which lies to the left of ℓ and, by the Hopf lemma (strong maximum principle), it follows that $u_N > v_N$ on this part of the boundary of D_h. This is the desired inequality since $v = u$ to the right of ℓ.

3. Numerical Results

We use a finite element method to calculate u_N^2 at a number of nodal points ϕ on ∂D_2, where ϕ is the angle between the radial line at the positive x-axis. Due to symmetry, it is sufficient to consider $0 \leq \phi \leq \pi$. The following tables give values for u_N^2 for various values of h and ϕ. The last row gives $\lambda(h)$ for different values of h.

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Table 1

Values for u_N^2

$a = 0.1 \qquad \lambda(0) = 10.98324859$

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ϕ	$h = 0.1$	$h = 0.3$	$h = 0.6$	$h = 0.8$
0°	0.18340156	0.08997194	0.03502936	0.00538128
15°	0.18586555	0.09354750	0.03875921	0.00792279
30°	0.19312533	0.10408993	0.04977909	0.01615017
45°	0.20478642	0.12105508	0.06745736	0.03118122
60°	0.22019869	0.14357017	0.09052611	0.05294918
75°	0.23846941	0.17048691	0.11728631	0.07901455
90°	0.25848609	0.20042583	0.14624640	0.10706183
105°	0.27895498	0.23176494	0.17645804	0.13678539
120°	0.29846292	0.26256868	0.20707716	0.16793719
135°	0.31556947	0.29053976	0.23653390	0.19964213
150°	0.32892971	0.31313057	0.26197403	0.22879649

ϕ				
165°	0.33743644	0.32789921	0.27954557	0.24990529
180°	0.34035750	0.33304454	0.28585725	0.25766770
$\lambda(h)$	10.51624800	8.76956649	6.91928150	6.21431318

Table 2
Values for u_N^2

$a = 0.3$ $\quad\quad \lambda(0) = 19.46950428$

	$h = 0.1$	$h = 0.3$	$h = 0.6$
ϕ			
0°	0.04651448	0.00601084	0.00006665
15°	0.05078040	0.00792264	0.00029224
30°	0.06389146	0.01432651	0.00162487
45°	0.08665951	0.02711431	0.00616138
60°	0.12001996	0.04901522	0.01734345
75°	0.16444947	0.08285892	0.03916871
90°	0.21927390	0.13049149	0.07481155
105°	0.28204163	0.19150347	0.12521694
120°	0.34820007	0.26211532	0.18784387
135°	0.41130766	0.33475001	0.25580537
150°	0.46389778	0.39885669	0.31827254
165°	0.49888764	0.44319924	0.36272535
180°	0.51117180	0.45907590	0.37887932
$\lambda(h)$	17.00607073	12.31240018	8.54494014

Table 3
Values for u_N^2

$$a = 0.6 \qquad \lambda(0) = 61.2854372$$

ϕ	$h = 0.1$	$h = 0.3$
0°	0.00010994	0.00000018
15°	0.00025775	0.00000144
30°	0.00101252	0.00002268
45°	0.00370221	0.00026580
60°	0.01190759	0.00195778
75°	0.03332159	0.00947178
90°	0.08086609	0.03287792
105°	0.17026477	0.08782665
120°	0.32267905	0.18896048
135°	0.49728793	0.33653240
150°	0.69311417	0.50402714
165°	0.84533543	0.64040281
180°	0.90307061	0.69330938
$\lambda(h)$	42.71463081	23.79696055

In all the cases above, u_N^2 increases in value as ϕ increases from zero to π, thereby confirming that $\dot{\lambda} < 0$ (see formula (2.6)). From the above tables we also note that for fixed a, $\lambda(h)$ is a decreasing function of h, and that $\lambda(h) < \lambda(0)$ for $h > 0$ thus confirming the Conjecture C.

REFERENCES

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2. T. Kato, *Perturbation theory for linear operators,* Springer Verlag, Berlin, 1966.